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THE TEACHING OF COLLEGE ALGEBRA.

By H. L. RIETZ, University of Illinois.

1. Introduction. The expression "College Algebra" is used in this paper to designate the algebra which should be taught to freshmen students who have met the entrance requirements of one and one half years of algebra in the secondary schools. This expression is here employed because it has come into rather general use in this country for the course given in the first college year, and not with any thought that it is deemed adequate algebraic training to be expected from a college graduate interested in mathematics. On this point, it is to be understood that a much more advanced course, which includes such subjects as the theory of linear dependence and the theory of matrices, should be taken late in the college course.

The question of the time at our disposal may considerably modify the character of the course in college algebra. In what follows, the writer assumes, as a minimum of time available for college algebra, that time which should be apportioned to algebra if trigonometry, algebra, and analytic geometry together are to constitute a course with five class periods per week through a college year.

2. Selection of material. The chief danger in the selection of material and presentation of college algebra is that it is likely to be a sort of scrap heap of disconnected or rather remotely connected topics, rather than an organized body of knowledge. While this is a danger, it is not a necessary characteristic of the course. The number of subjects classed under college algebra is usually too large for the time allotted to the course. The presentation of so many subjects is due, in the main, to the call from branches of mathematics higher up for a working knowledge on the part of the student of various algebraic methods; and, certainly, one criterion for judging the importance of a subject is its use in other branches of mathematics. It is not undesirable to treat a large number of topics, provided the time is available, and if there are a few underlying principles or a method of attack by which the topics are unified or brought into relation with each other. In fact, it is familiarity with a large variety of topics and their inter-relations that should be sought in elementary college courses in mathematics.

3. Unifying elements of the course. It appears that there are at least two ways by which topics of a branch of mathematics may be educationally unified; that is, so presented or organized that the student feels that the branch in question is a connected body of knowledge. For example, in the study of Euclid, a student is surely impressed with the logical element. That is to say, the body of geometric facts seems to him to be unified by definitions and axioms. In the study of analytic geometry, the student is hardly impressed with logical considerations at all, but the subject should be as thoroughly unified in the method of presentation by reference to the idea of correspondence between an equation and a curve, as is Euclid by logical considerations.

College algebra should be unified, to some extent, by logical considerations; but, in the opinion of the writer, it should be presented so that the student will feel that it is even more unified by reference to a few central facts, as in the case of analytic geometry. While secondary school algebra should perhaps be presented, in the main, by generalizations from particular numerical cases, in college algebra, the explicit statements of definitions and assumptions on which to base proofs, and the deduction from these statements of some principles of algebra, should constitute one kind of unifying basis. This does not imply, by any means, that the course includes a critical study of fundamentals. Freshmen students are not prepared for such a study, but they are prepared to appreciate that principles of algebra, as well as the propositions of geometry, can be established by logical arguments. The number of stated assumptions should be too many rather than too few, provided the assumptions are readily accepted by the student and are mathematically sound.

In §§7—13 of this paper, it is the purpose to indicate briefly how some of the subjects of college algebra, that are likely to appear isolated, can be unified in method of presentation by much reference to the *equation as a condition to be satisfied,* and to the *function as a variable whose changes are to be traced.*

4. Relation to secondary algebra. The view has been frequently expressed by writers on mathematical education that the mathematics of the college is not well correlated with the mathematics as taught at present in the secondary schools, and that the ideas of the best trained mathematicians have little influence on secondary instruction. To meet the need thus felt for a closer correlation of ideas, one of the aims of the course in college algebra should be to bridge the gap between the ideas emphasized in the high school and in the college by treating topics of secondary algebra in the college course more in the light of higher mathematics, and with greater stress on logical considerations than is, in general, feasible in the high school.

5. Nature of exercises and problems. In connection with the establishment of each important principle, there should be given, whenever feasible, illustrative problems so connected with the experience of the pupil as

to make the principle appear of real value on account of its applications. That such problems exist has been shown, but the teacher can do a great service by adding to the supply existent.

6. A high school view of equations. Almost invariably, the student who has completed merely the algebra offered in our secondary schools, regards the equation solely as an equality containing unknowns to be found. To find these unknowns, the student has learned to perform certain mechanical processes on the members of the equation—these processes being suggested by the simple operations of arithmetic and not really established for the more general numbers of algebra. The validity of the processes under any conditions is hardly ever called into question in the secondary algebra. Whether substantial improvement in results bearing on this point are to be expected from secondary schools in the near future is beyond the purpose of this paper. But, at present, surely, the condition obtains that in so far as the equation is an object of thought for the entering freshman, he regards it almost solely with reference to finding the unknowns it contains.

7. A second view of equations. In college algebra, the chief emphasis should be placed on the equation as a *condition* to be satisfied rather than as something containing unknowns to be found. To show that equations are not thus regarded by students entering college, and that this fact leads to some difficulties, let me say that I find every year in discussing the factor theorem that some of the superior students argue as follows:

"If a is a root of $f(x)=0$, this means that the unknown x is a, and to say that $f(x)$ is divisible by $x-a$ is equivalent to saying that zero is a divisor of $f(a)$. But this is contrary to the fundamental statement that division by zero is excluded from mathematical operations."

To lead the student to correct this fallacy, and to substitute for it correct views on such points is one of the difficult tasks of the instructor in college algebra.

8. Idea of functionality. In order to appreciate fully and deeply the significance of satisfying even simple equations such as $y=3x+4$ and $y=4x-6$, the graph is almost indispensable. With the graph comes the idea of functionality; and, in my opinion, a treatment of the graphs of linear and quadratic functions should precede the review of the linear and quadratic equations in this course.

To get some idea of the recent tendency to appeal to geometrical representations in college algebra, I have just examined eight text books in college algebra that I happen to have on hand—four of these books have been written within the past five years, and four others from ten to nineteen years ago. I find that the newer set of books are adorned with curves and figures to the extent of from twenty-four to thirty-nine cuts while the older books contain from none to nineteen cuts—most of the latter being given to represent complex numbers. While I recognize that the character of the text is not, in general, identical with the course, I think the change in the

character of text books is a significant measure of the tendency to make graphs and the idea of functionality more prominent than formerly.

9. Determinants. In expressing solutions of linear equations, and in placing conditions on coefficients of various equations to correspond to certain facts in regard to the roots, the determinant notation is such an important mechanical aid in giving elegance of form to the work that the elements of determinants should be presented for use in this course and in analytical geometry, rather than as an isolated subject almost entirely separated from the equation.

10. Extensions of the number concept. In seeking numbers to satisfy equations, there is the opportunity to interest the student in extensions of the number concept, and to treat in a very elementary way irrational and complex numbers. The writer has been much impressed by the interest of his students in making extensions of the number concept appear necessary to meet the demands of the equation. This method of extending the number concept may, of course, give a wrong view unless it is pointed out that some irrational numbers, such as π for example, are not roots of an equation with rational coefficients.

11. Theory of equations. In the time at our disposal, we can perhaps expect only that the solution for real roots of special numerical equations be reached as a goal in the theory of equations. This problem is not only of value for its direct applications, but also because it brings before the student a process of successive approximation of much importance as an illustration of general methods of successive approximation. However, the points to be kept most prominently before the student in the theory of equations are that we are regarding $f(x)=0$ as a relation to be satisfied, and $f(x)$ as a function whose changes in value concern us.

11. Logarithms. In the plotting of graphs of functions, expressions of sufficient complexity should be given to bring the student to seek the best methods of numerical evaluation. This calls for the use of logarithms, which, it is generally hoped, are used in calculations in the secondary course, but are often not used in such a course.

12. Limits and infinite series. There seems to exist considerable difference of opinion among instructors of college mathematics as to the advisability of presenting the elements of the theory of limits and infinite series in a freshman course. If the subject of infinite series is to be included in the course, it can be very effectively presented with emphasis on the fact that the first n terms of a series is a function which changes as n takes values 1, 2, 3, 4, ..., and that it is the limit of this function as n grows beyond any fixed bound in which we are interested. In this way, the subject of infinite series is referred to one of the unifying elements of the course.

To write the generating function from a few terms of a series is a most valuable exercise in leading students to grasp what may be called algebraic form. Herein is the recognition of algebraic law from particular

cases. Even the simple direct process of expanding $\sum_{n=1}^{n=\infty} u_n$, where u_n is some function of n such as $\frac{1}{(2n-1)\,!}$, forms a sort of prelude to the use of the summation sign in the calculus.

The value of the subject of infinite series does not therefore rest solely either on the practical ground that the student needs to test the convergence of special series that arise in analysis, nor on the ground that it is a suitable field for the growth of the limit concept.

The writer does not find himself in agreement with those who hold that it is a more natural order of presentation to defer the study of infinite series until late in the course in calculus. He questions whether this position is tenable, especially if series are presented with a considerable use of geometric representation. That is to say, if the sums of n terms of a few carefully selected series are plotted as ordinates to correspond to $n=1, 2, 3, \ldots$ used as abscissas, so that the function idea is prominent, the process of approach to a limit makes a valuable appeal to the student's common sense. While the answer to the question of including this subject in a freshman course may be changed by the amount of time allotted to the course in calculus, it is the opinion of the writer that a good deal is gained by presenting to freshmen students in considerable detail, the elementary tests for the convergence and divergence of series, if fewer than five hours per week through a school year are given to calculus.

Wherever convergence of series is studied, the treatment should be marked by precision of statements. The arguments should be in harmony with the unifying principles of this course in algebra; that is, assumptions on which proofs are based should be explicitly stated.

13. Summary. In this paper, the writer has aimed to consider especially certain subjects whose place in college algebra is sometimes called into question, and to direct attention to points which should be particularly emphasized.

It seems that while the course in college algebra should be unified, to some extent, by logical considerations, it should be unified in method of presentation by much reference to 1) the equation as a relation to be satisfied, 2) the function as a variable whose changes in value are to be traced.

It is by referring much to these central and closely related ideas, and by the introduction of more practical problems (§ 5) that we may well expect to improve our teaching of college algebra.